Lecture 7

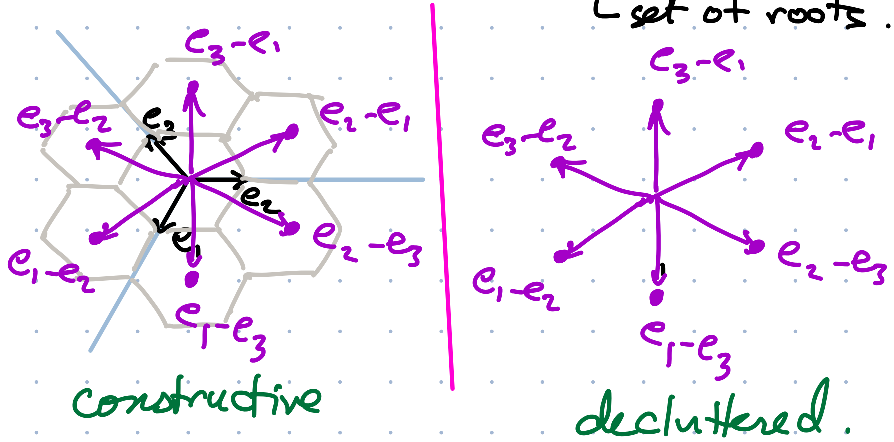
CARTAN SUBALGEBRA, WEYL GROUP, & ROOT SYSTEM

Last time: $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$ defined $\mathfrak{h} = \dots$ $\mathfrak{a} = \dots$ got $\underbrace{\Phi}_{\text{set of roots}} \subset \overbrace{\mathfrak{a}^*}^{\mathbb{R}}$ or $\subset \overbrace{\mathfrak{h}^*}^{\mathbb{C}}$.

Picture for $\mathfrak{sl}_3\mathbb{C}$.

$$\mathfrak{a}^* = \mathbb{R}^3_{a_1 a_2 a_3}/(1,1,1)$$
$$\cong \{a_1 + a_2 + a_3 = 0\}$$

$$\Phi = \{e_i - e_j,\ i \neq j\}$$
$$= \{ [(1,-1,0)], [(-1,1,0)], [(0,1,-1)], [(0,-1,1)], [(-1,0,1)], [(1,0,-1)] \}$$



The root system of $\mathfrak{sl}_3\mathbb{C}$.

Generalizing. $\mathfrak{g}$ semisimple Lie alg $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalg if ① it is a subalgebra of $\mathfrak{g}$, ② is abelian, ③ $ad_t : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable $\forall\, t \in \mathfrak{h}$, and it is not properly contained in another subalg satisfying ①-③.

Thm. Such $\mathfrak{h}$ exist and any two Cartan subalgebras are related by a Lie algebra automorphism of $\mathfrak{g}$ (in fact an inner one).

Thus get $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ for subset $\Phi \subset \mathfrak{h}^*$ (finite)

Note $\mathfrak{h} \subset \ker(ad_{\mathfrak{h}})$ clearly, but we have equality as $\mathfrak{h}$ is maximal.

First properties of the roots:

Thm. 1) $\alpha \in \Phi$ iff $-\alpha \in \Phi$
2) $\mathfrak{g}_\alpha$ is 1-dimensional 3) If $\alpha \in \Phi$ and $k\alpha \in \Phi$ for $k \in \mathbb{Z}$ then $k = \pm 1$.
4) Φ spans $\mathfrak{h}^*$, and $V = \mathbb{R}$-span of Φ is a real form of $\mathfrak{h}^*$.

Geometry of the roots.

$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$: $B(x,y) = \operatorname{tr}(\underbrace{ad_x \circ ad_y}_{\text{a } \dim \mathfrak{g} \times \dim \mathfrak{g} \ \mathbb{C} \text{ matrix.}})$ symmetric $\mathbb{C}$-bilinear.

Semisimple $\Leftrightarrow$ B nondegenerate (Cartan's criterion)

Thm. 1) $B|_{\mathfrak{h} \times \mathfrak{h}}$ is nondegenerate, hence induces iso $\mathfrak{h}^* \to \mathfrak{h}$ and a symmetric bilinear form B^* on $\mathfrak{h}^*$

2) let $\langle , \rangle$ denote the restr of B^* to $V = \mathbb{R}$-span roots. Then $\langle , \rangle$ is $\mathbb{R}$-valued and pos def.

3) $\forall\, \alpha, \beta \in \Phi$, $\dfrac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

4) $\forall\, \alpha \in \Phi$, the map $s_\alpha : V \to V$ by $s_\alpha(\varphi) = \varphi - \dfrac{2\langle \varphi, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$ preserves Φ ($s_\alpha(\Phi) = \Phi$).

$s_\alpha|_{\mathbb{R}\alpha} = -1$ and $s_\alpha|_{(\mathbb{R}\alpha)^\perp} = 1$. These generate a finite group $\subset O(V, \langle , \rangle)$, $\subset \mathrm{Sym}_\Phi$.

Reduced root system

Def. A finite-dim'l real inner product space $(V, \langle , \rangle)$ and finite set $\Phi \subset V$ s.t.

1) Φ spans V
2) $\forall\, \alpha \in \Phi$, s_α preserves Φ
3) $\dfrac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ $\forall\, \alpha, \beta \in \Phi$.
4) $\alpha \in \Phi \Rightarrow 2\alpha \notin \Phi$

Note. Can scale $\langle , \rangle$ by positive constant and cond will still be satisfied. This is considered an iso of root systems.

Cor. $\mathfrak{g}$ gives a reduced abstract root system, using Cartan.

Thm: For semisimple $\mathfrak{g}$, $\mathfrak{g}'$, we have $\mathfrak{g} \cong \mathfrak{g}'$ iff the root systems are isomorphic ($T : V \to V'$ isom $T(\Phi) = \Phi'$)

Thm. $\mathfrak{g}$ simple $\iff$ $(V, \langle,\rangle, \Phi)$ irred, i.e. can't write

$$V = V_1 \oplus V_2 \qquad V_1 \perp V_2$$
$$\Phi = \Phi_1 \cup \Phi_2 \qquad \underset{\Phi_1}{\overset{\cup}{}} \qquad \underset{\Phi_2}{\overset{\cup}{}}$$

Classifying irreducible root systems (hence $\mathbb{C}$ simp Lie alg)

$(V, \langle,\rangle)$ irred root system.

We want to cut down from Φ (which has a lot of symmetry) to a smaller set that's lin indep.

Introduce a notion of positivity on V: Pick a basis $v_1 \cdots v_r$ and say $\varphi > 0$ if the first nonzero coef of φ rel to $v_1 \cdots v_r$ is positive. Thus if $\varphi \neq 0$, then either $\varphi > 0$ or $\varphi < 0$.

Let $\Phi^+ \subset \Phi$ be the set of positive roots. ($\Phi = \Phi^+ \cup -\Phi^+$)

$\alpha \in \Phi^+$ is called simple if it is not a sum of two other positive roots.

Let Δ denote the set of simple roots.

Thm. Δ is a basis of V. (the rank of $(V, \langle\rangle)$).
$\mathbb{Z}$-span of Δ contains Φ.

$$A_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}$$ i, j run over the set of simple roots $\rightarrow$ Cartan matrix.
INTEGER

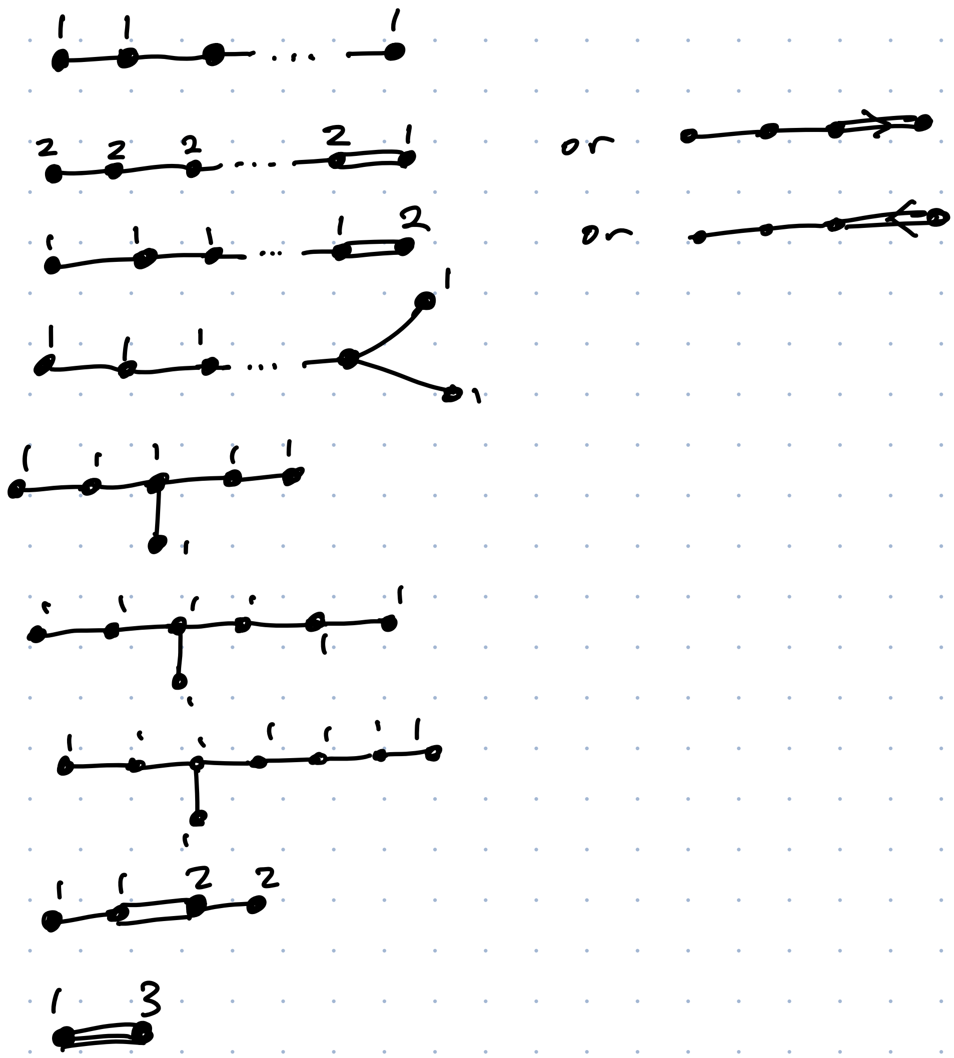
Dynkin diagram. Graph w/ vertex set Δ, $A_{ij}A_{ji}$ edges from α_i to α_j, and where each vertex has a weight on it prop to $\langle \alpha_i, \alpha_i \rangle$.

(Note $\dfrac{\langle \alpha, \alpha \rangle}{\langle \beta, \beta \rangle} = \dfrac{\langle \alpha, \alpha \rangle}{2\langle \alpha, \beta \rangle} \dfrac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} = A_{ij}/A_{ji}$ is rational)

Thm. The Dynkin diag is uniquely det (up to graph iso / weight scaling) by the iso class of the root system. (does not depend on the ordering)

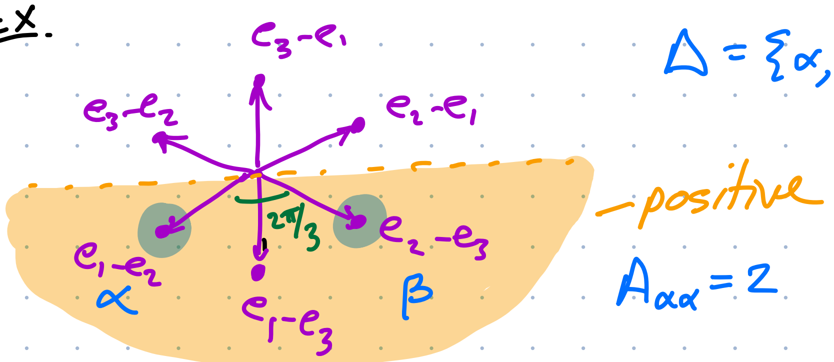
Thm. The possible dynkin diag obtained from irred root sys:

A_ℓ, B_ℓ, C_ℓ, D_ℓ, E_6, E_7, E_8, F_4, G_2



arrow toward shorter root.

Ex.



$\Delta = \{\alpha, \beta\}$ $\quad \langle \alpha, \alpha\rangle = \langle \beta,\beta\rangle = 1$, say

$\langle \alpha, \beta\rangle = \cos\frac{2\pi}{3} = -\frac{1}{2}$

$A_{\alpha\alpha} = 2 \quad A_{\alpha\beta} = \frac{2\cdot(-\frac{1}{2})}{1} = -1$

$A_{\beta\beta} = 2 \quad A_{\beta\alpha} = \frac{2\cdot(-\frac{1}{2})}{1} = -1$

$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \rightsquigarrow$